

A Generalization of Kneser's Conjecture

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Abstract

We investigate some coloring properties of Kneser graphs. A star-free coloring is a proper coloring $c : V(G) \rightarrow \mathbb{N}$ such that no path with three vertices may be colored with just two consecutive numbers. The minimum positive integer t for which there exists a star-free coloring $c : V(G) \rightarrow \{1, 2, \dots, t\}$ is called the star-free chromatic number of G and denoted by $\chi_s(G)$. In view of Tucker-Ky Fan's lemma, we show that $\chi_s(\text{KG}(n, k)) = 2\chi(\text{KG}(n, k)) - 2 = 2n - 4k + 2$ provided that $n \leq \frac{8}{3}k$. This gives a partial answer to a conjecture of [12]. Moreover, we show that for any Kneser graph $\text{KG}(n, k)$ we have $\chi_s(\text{KG}(n, k)) \geq \max\{2\chi(\text{KG}(n, k)) - 10, \chi(\text{KG}(n, k))\}$ where $n \geq 2k \geq 4$. Also, we conjecture that for any positive integers $n \geq 2k \geq 4$ we have $\chi_s(\text{KG}(n, k)) = 2\chi(\text{KG}(n, k)) - 2$.

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1 Introduction

The local coloring of a graph G , defined in [2] and [3], is a proper coloring $c : V(G) \rightarrow \mathbb{N}$ such that no path with three vertices and no triangle may be colored with just two and three consecutive numbers respectively. In other words, for any set $S \subseteq V(G)$ with $2 \leq |S| \leq 3$, there exist two vertices $u, v \in S$ such that $|c(u) - c(v)| \geq m_S$, where m_S is the number of edges of the induced subgraph $G[S]$. The maximum color assigned by a local coloring c to a vertex of G is called the value of c and is denoted by $\chi_l(G, c)$. The local chromatic number of G is $\chi_l(G) = \min_c \chi_l(G, c)$, where the minimum is taken over all local colorings c of G . If $\chi_l(G) = \chi_l(G, c)$, then c is called a minimum local coloring of G .

A star-free coloring is a proper coloring $c : V(G) \rightarrow \mathbb{N}$ such that no path with three vertices may be colored with just two consecutive numbers. The minimum natural number t for which there exists a star-free coloring $c : V(G) \rightarrow \{1, 2, \dots, t\}$ is called the star-free chromatic number of G and denoted by $\chi_s(G)$. Also, if $c : V(G) \rightarrow \{1, 2, \dots, t\}$ is a star-free coloring of G and $t = \chi_s(G)$, then c is called a minimum star-free coloring. Note that a local coloring is a star-free coloring with one more requirement (no triangle may be colored with just three consecutive numbers).

Hereafter, the symbol $[n]$ stands for the set $\{1, \dots, n\}$. Assume that $n \geq 2k$. The Kneser graph $\text{KG}(n, k)$ is the graph with vertex set $\binom{[n]}{k}$, in which A is connected to B if and only if $A \cap B = \emptyset$. For a subset $X \subseteq [n]$ denote by $\binom{X}{k}$ the subgraph induced by the collection of all k -subsets of X in $\text{KG}(n, k)$. It was conjectured by Kneser [7] in 1955, and proved by Lovász [8] in 1978, that $\chi(\text{KG}(n, k)) = n - 2k + 2$.

Let $c : V(G) \rightarrow \mathbb{N}$ be a proper coloring of a graph G . If we replace the color i by $2i - 1$ for any positive integer $1 \leq i \leq \chi(G)$, then we obtain a local coloring for G .

Lemma A. [3] *For any graph G we have*

$$\chi(G) \leq \chi_l(G) \leq 2\chi(G) - 1.$$

In view of definition of star-free chromatic number, one can deduce that $\chi(G) \leq \chi_s(G) \leq \chi_l(G) \leq 2\chi(G) - 1$.

In [12], the local chromatic number of Kneser graphs was studied and the local chromatic numbers of the Kneser graphs $K(2k+1, k)$ and $\text{KG}(n, 2)$ were determined. Also, it was shown that for any positive integers n and k with $n \geq 2k$, $\chi_l(\text{KG}(n, k)) \leq 2\chi(\text{KG}(n, k)) - 2 = 2n - 4k + 2$. To see this, for any $1 \leq i \leq n - 2k + 1$ set

$$C_{2i-1} \stackrel{\text{def}}{=} \{A \in V(\text{KG}(n, k)) : \{1, 2, \dots, i\} \cap A = \{i\}\},$$

$$C_{2n-4k+2} \stackrel{\text{def}}{=} \{A \in V(\text{KG}(n, k)) : \{1, 2, \dots, n - 2k + 1\} \cap A = \emptyset\}.$$

Now, it is readily seen that the aforementioned partition is a local coloring, consequently, $\chi_s(\text{KG}(n, k)) \leq \chi_l(\text{KG}(n, k)) \leq 2\chi(\text{KG}(n, k)) - 2 = 2n - 4k + 2$.

In [12], it was conjectured that for any positive integers $n \geq 2k \geq 4$ we have $\chi_l(\text{KG}(n, k)) = 2\chi(\text{KG}(n, k)) - 2 = 2n - 4k + 2$.

Conjecture A. [12] *For every positive integers n and k with $n \geq 2k \geq 4$, we have $\chi_l(\text{KG}(n, k)) = 2\chi(\text{KG}(n, k)) - 2 = 2n - 4k + 2$.*

In view of Lemma A, we obtain $\chi(\text{KG}(n, k)) \geq \frac{\chi_l(\text{KG}(n, k))}{2} + 1$. Hence, the aforementioned conjecture can be considered as a generalization of Kneser's conjecture.

Theorem A. [12] *Let n and k be positive integers where $n \geq 2k \geq 4$. If $\text{KG}(n, k)$ has a minimum local coloring with a color class of size at least $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2$, then $\chi_l(\text{KG}(n, k)) = \chi_l(\text{KG}(n-1, k)) + 2$.*

In the proof of the aforementioned theorem, the authors do not employ the triangle condition (no triangle may be colored with three consecutive numbers). Consequently, the above theorem holds for star-free chromatic number as well. Here, we sketch the proof of Theorem A. Let $c : V(\text{KG}(n, k)) \rightarrow \{1, 2, \dots, \chi_s(\text{KG}(n, k))\}$ be a star-free coloring. Set $C_i \stackrel{\text{def}}{=} c^{-1}(i)$ for any $1 \leq i \leq \chi_s(\text{KG}(n, k))$. Also, assume that C_j is a color class with size at least $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2$. The Hilton and Milner theorem [6] says that if I is an independent set of $\text{KG}(n, k)$ of size at least $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2$, then for some $i \in [n]$ we have $\cap_{A \in I} A = \{i\}$. Hence, there exists some $i \in [n]$ such that $\cap_{A \in C_j} A = \{i\}$. Now it is easy to check that if $j \geq 2$ (resp. $j \leq n - 1$), then $i \in \cap_{A \in C_{j-1}} A$ (resp. $i \in \cap_{A \in C_{j+1}} A$). By the above claim the vertices with the colors $j - 1, j, j + 1$ induce an empty subgraph. Without loss of generality, suppose $i = n$. Now we define the coloring $c' : V(\text{KG}(n - 1, k)) \rightarrow \{1, 2, \dots, \chi_s(\text{KG}(n, k)) - 2\}$ as follows.

$$c'(A) \stackrel{\text{def}}{=} \begin{cases} c(A) & \text{if } c(A) \leq j - 2 \\ c(A) - 2 & \text{if } c(A) \geq j + 2. \end{cases}$$

One can check that c' is a star-free coloring for $\text{KG}(n-1, k)$, hence, $\chi_s(\text{KG}(n-1, k)) \leq \chi_s(\text{KG}(n, k)) - 2$. On the other hand, any star-free coloring of $\text{KG}(n-1, k)$ with t colors can be extended to a star-free coloring of $\text{KG}(n, k)$ with $t+2$ colors. Consequently, $\chi_s(\text{KG}(n, k)) = \chi_s(\text{KG}(n-1, k)) + 2$.

Corollary 1. *Let $n \geq 2k \geq 4$ be positive integers. If $\text{KG}(n, k)$ has a minimum star-free coloring with a color class of size at least $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2$, then $\chi_s(\text{KG}(n, k)) = \chi_s(\text{KG}(n-1, k)) + 2$.*

Section 2 presents some preliminaries. In Section 3, in view of Tucker-Ky Fan's lemma, we show that for any Kneser graph $\text{KG}(n, k)$ we have $\chi_s(\text{KG}(n, k)) \geq \max\{2\chi(\text{KG}(n, k)) - 10, \chi(\text{KG}(n, k))\}$ where $n \geq 2k \geq 4$. Moreover, we show that if k is sufficiently large then $\chi_l(\text{KG}(2k+t, k)) = \chi_s(\text{KG}(2k+t, k)) = 2\chi(\text{KG}(2k+t, k)) - 2 = 2t + 2$. This gives a partial answer to Conjecture A.

2 Borsuk-Ulam Theorem and its Generalizations

The Lovász's proof [8] of the Kneser's conjecture was an outstanding application of topological methods in 1978. In fact, the proof of Lovász is based on the Borsuk-Ulam theorem. The Borsuk-Ulam theorem says that if $f : S^n \rightarrow \mathbb{R}^n$ is a continuous mapping from the unit sphere in \mathbb{R}^{n+1} into \mathbb{R}^n , then there exists a point $x \in S^n$ where $f(x) = f(-x)$, that is, some pair of antipodal points has the same image. There are several different equivalent versions, various different proofs, several extensions and generalizations, and many interesting applications for the Borsuk-Ulam theorem that mark it as a great theorem, see [4, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19].

Let S^n denote the n -sphere, i.e., $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$. Assume that T is a triangulation of S^n . The triangulation T is termed antipodally symmetric around the origin, if σ is a simplex in T , then $-\sigma$ is also a simplex in T . Tucker's lemma is a combinatorial analogue of the Borsuk-Ulam theorem with several useful applications, see [9, 10].

Lemma B. (*Tucker's lemma.* [17]) *Let T be a symmetric triangulation of the n -sphere S^n where n is a positive integer. Assume that each vertex u of T is assigned a label $\lambda(u) \in \{\pm 1, \pm 2, \dots, \pm n\}$ such that λ is an antipodal map, i.e., $\lambda(-u) = -\lambda(u)$ for any vertex u of T . Then some pair of adjacent vertices of T have labels that sum to zero.*

Another interesting generalization of Borsuk-Ulam theorem is Ky Fan's lemma [4], which generalizes the LusternikSchnirelmann theorem which is the version of the Borsuk-Ulam theorem involving a cover of the n -sphere by $n+1$ sets, all open or all closed. Just like the Borsuk-Ulam theorem it has several equivalent forms, see [4].

Lemma C. (*Ky Fan's lemma.*) *Let n and k be two arbitrary positive integers. Assume that k closed subsets (resp. open subsets) F_1, F_2, \dots, F_k of the n -sphere S^n cover S^n and also no one of them contains a pair of antipodal points. Then there exist $n+2$ indices l_1, l_2, \dots, l_{n+2} , such that $1 \leq l_1 < l_2 < \dots < l_{n+2} \leq k$ and*

$$F_{l_1} \cap -F_{l_2} \cap F_{l_3} \cap \dots \cap (-1)^{n+1} F_{l_{n+2}} \neq \emptyset.$$

where $-F_i$ denotes the antipodal set of F_i . In particular, $k \geq n + 2$.

This lemma has useful applications in graph colorings and provides useful information about coloring properties of Kneser graphs, see [11, 15, 16, 20]. For instance, it was shown that the circular chromatic number and the chromatic number of the Kneser graph $\text{KG}(n, k)$ are equal provided that n is even, see [11, 15]. Moreover, we can consider the subcoloring theorem as an interesting application of Ky Fan's lemma, see [18, 19]. This theorem states if c is a proper coloring of the Kneser graph $\text{KG}(n, k)$ with m colors, then there exists a multicolored complete bipartite graph $K_{\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor}$ with $r \stackrel{\text{def}}{=} \chi(\text{KG}(n, k))$ such that the r different colors occur alternating on the two sides of the bipartite graph with respect to their natural order. The subcoloring theorem has been generalized for general Kneser graphs in [16].

In [4], Fan introduced a generalization of Tucker's lemma which is called Tucker-Ky Fan's lemma.

Lemma D. (*Tucker-Ky Fan's lemma.*) *Let T be a symmetric triangulation of S^n and m be a fixed positive integer. Also, assume that each vertex u of T is assigned a label $\lambda(u) \in \{\pm 1, \pm 2, \dots, \pm m\}$ such that λ is an antipodal map and furthermore labels at adjacent vertices do not sum to zero. Then there are an odd number of n -simplices whose labels are of the form $\{a_0, -a_1, \dots, (-1)^n a_n\}$, where $1 \leq a_0 < a_1 < \dots < a_n \leq m$. In particular $m \geq n + 1$.*

Now, we introduce a special triangulation of S^{n-1} . Let e_1, e_2, \dots, e_n be the vectors of the standard orthonormal basis of R^n (e_i has a 1 at position i and 0's elsewhere). Define a simplicial complex C^{n-1} (cross polytope) as follows. The vertex set of C^{n-1} is equal to $\{\pm e_1, \pm e_2, \dots, \pm e_n\}$. Also, a subset $F \subseteq \{\pm e_1, \pm e_2, \dots, \pm e_n\}$ forms the vertex set of a face of the cross polytope if there is no $i \in [n]$ with both $e_i \in F$ and $-e_i \in F$.

Set $V_n \stackrel{\text{def}}{=} \{-1, 0, 1\}^n$. Consider a partial ordering \leq on V_n that for any $u, v \in V_n$ we have $u \leq v$ if $u_i \leq v_i$ for any $1 \leq i \leq n$ where $0 \leq -1$ and $0 \leq 1$. Now, we introduce a symmetric triangulation of S^{n-1} , say K , whose vertex set can be identified with the vectors of $V_n \setminus \{(0, 0, \dots, 0)\}$.

Let K be the first barycentric subdivision of C^{n-1} . Thus, the vertices of K are centers of gravity of the simplices of C^{n-1} and the simplices of K correspond to chains of simplices of C^{n-1} under inclusion. In fact, the vertex set of K can be identified with $V_n \setminus \{(0, 0, \dots, 0)\}$. Moreover, a simplex of K can be identified uniquely with a chain in the set $V_n \setminus \{(0, 0, \dots, 0)\}$ under \leq . Also, one can see that K is a symmetric triangulation of S^{n-1} which meets conditions in Tucker-Ky Fan's lemma. Hence, to use the Tucker-Ky Fan's lemma, it is sufficient to define a labeling $\lambda : V_n \setminus \{(0, 0, \dots, 0)\} \rightarrow \{\pm 1, \pm 2, \dots, \pm m\}$ which has the antipodal property, i.e., for any nonzero $v \in V_n$ we have $\lambda(-v) = -\lambda(v)$, and furthermore, for any $u, v \in V_n \setminus \{(0, 0, \dots, 0)\}$ where $u \leq v$ or $v \leq u$, we have $\lambda(u) + \lambda(v) \neq 0$, that is, labels at adjacent vertices do not sum to zero.

3 Star-Free Chromatic Number of Kneser Graphs

It was shown in [12] if $n \geq 2k^3 - 2k^2$, then $\chi_l(\text{KG}(n, k)) = \chi_l(\text{KG}(n - 1, k)) + 2$.

Theorem 1. *Let $n \geq 2k \geq 4$ be positive integers where $n \geq 2k^3 - 2k^2 - 2k + 4$. Then $\chi_s(\text{KG}(n, k)) = \chi_s(\text{KG}(n-1, k)) + 2$.*

Proof. Let c be a minimum star-free coloring of $\text{KG}(n, k)$. In view of Corollary 1, it is sufficient to show that there exists a color class of c with size at least $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2$. Note that $\chi_s(\text{KG}(n, k)) \leq 2\chi(\text{KG}(n, k)) - 2$ and also it was shown in [5] that if $n \geq 2k^2(k-1)$, then $\frac{\binom{n}{k}}{2(n-2k+2)} \geq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2$. Hence, for $k = 2$ the assertion follows. Now assume that $k \geq 3$. By double counting we have

$$\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 2 \leq k \binom{n-2}{k-2}. \quad (1)$$

Hence, if we show that there exists a color class of c with size at least $k \binom{n-2}{k-2}$, then $\chi_s(\text{KG}(n, k)) = \chi_s(\text{KG}(n-1, k)) + 2$. For any positive integers a and b where $a \geq 2k-2$ and $b \leq 2k-3$ we have

$$a(a-b-2k+1) \leq \left(a - \frac{b+2k-1}{2}\right)^2 \leq (a-b)(a-b-1). \quad (2)$$

Let $b \leq 2k-3$ and $a \geq 2k^3 - 2k^2$. Set $n \stackrel{\text{def}}{=} a-b$. In view of (2), it is readily seen that if $n \geq 2k^3 - 2k^2 - 2k + 3$, then

$$(2k^3 - 2k^2)(n-2k+1) \leq a(a-b-2k+1) \leq (a-b)(a-b-1) = n(n-1).$$

Consequently,

$$\frac{\binom{n}{k}}{k \binom{n-2}{k-2}} = \frac{n(n-1)}{k^2(k-1)} \geq 2(n-2k+1) = 2\chi(\text{KG}(n, k)) - 2 \geq \chi_s(\text{KG}(n, k)).$$

Therefore, there is a color class of c with size at least $k \binom{n-2}{k-2}$, as desired. \blacksquare

Remark 1. In [1], the inequality (1) has been applied to show that if $n \geq 2k^3 - 2k^2 - 2k + 4$, then the circular chromatic number and the chromatic number of the Kneser graph $\text{KG}(n, k)$ are equal (for definition and more on circular chromatic number see [20]).

Matoušek [10] introduced an interesting proof for the Kneser conjecture by using Tucker's lemma. Similarly, in view of Tucker-Ky Fan's lemma, one can show that for any positive integers $n \geq 2k$, we have $\chi_s(\text{KG}(n, k)) \geq 2\chi(\text{KG}(n, k)) - 10 = 2n - 4k - 6$.

Theorem 2. *For every positive integers $n \geq 2k \geq 4$, we have $\chi_s(\text{KG}(n, k)) \geq \max\{2\chi(\text{KG}(n, k)) - 10, \chi(\text{KG}(n, k))\}$.*

Proof. Let k be a fixed positive integer. In view of Theorem 1, it is sufficient to show that $\chi_s(\text{KG}(n, k)) \geq 2\chi(\text{KG}(n, k)) - 10$ provided that $n < 2k^3 - 2k^2 - 2k + 4$. Thus, assume that $2k \leq n \leq 2k^3 - 2k^2 - 2k + 3$. Also, on the contrary, suppose that $\chi_s(\text{KG}(n, k)) \leq 2\chi(\text{KG}(n, k)) - 11 = 2n - 4k - 7$.

Suppose that c is a star-free coloring of the Kneser graph $\text{KG}(n, k)$ with $2n-4k-7$ colors. Suppose that the colors are numbered $2k-1, 2k, \dots, m = 2n-2k-9$. Now, we introduce a labeling $\lambda : V_n \setminus \{(0, 0, \dots, 0)\} \rightarrow \{\pm 1, \pm 2, \dots, \pm m\}$. Consider an arbitrary linear ordering \leq on power set of $[n]$ that refines the partial ordering according to size, that is, if $|A| < |B|$ then $A < B$.

Let $w = (w_1, w_2, \dots, w_n) \in V_n \setminus \{(0, 0, \dots, 0)\}$. To define $\lambda(w)$, we consider the ordered pair $(P(w), N(w))$ of disjoint subsets of $[n]$ defined by

$$P(w) \stackrel{\text{def}}{=} \{i \in [n] : w_i = +1\} \text{ and } N(w) \stackrel{\text{def}}{=} \{i \in [n] : w_i = -1\}.$$

We consider two cases. If $|P(w)| + |N(w)| \leq 2k-2$ (Case I) then we set

$$\lambda(w) \stackrel{\text{def}}{=} \begin{cases} |P(w)| + |N(w)| & \text{if } P(w) \geq N(w) \\ -|P(w)| - |N(w)| & \text{if } P(w) < N(w). \end{cases}$$

Now assume that $|P(w)| + |N(w)| \geq 2k-1$ (Case II). Note that if $|P(w)| + |N(w)| \geq 2k-1$ then at least one of $P(w)$ and $N(w)$ has size at least k . If $P(w) \geq N(w)$ (resp. $P(w) < N(w)$) we define $\lambda(w) = t$ (resp. $\lambda(w) = -t$) where t is the largest positive integer such that there exist two distinct k -subsets $A, B \subseteq P(w)$ (resp. $A, B \subseteq N(w)$) where $c(A) = c(B) = t$, otherwise, t is the largest positive integer such that there exists a k -subset $A \subseteq P(w)$ (resp. $A \subseteq N(w)$) where $c(A) = t$.

It is a simple matter to check that the labeling λ is well-defined and that it has the antipodal property, i.e., for any nonzero $v \in V_n$ we have $\lambda(-v) = -\lambda(v)$. Furthermore, labels at adjacent vertices do not sum to zero. Hence, in view of Tucker-Ky Fan's lemma, there exist an $(n-1)$ -simplex (a chain of length n in V_n), say σ , whose labels are of the form $\{a_0, -a_1, \dots, (-1)^{n-1}a_{n-1}\}$, where $1 \leq a_0 < a_1 < \dots < a_{n-1} \leq m = 2n-2k-9$. Let $V(\sigma) = \{v_1, v_2, \dots, v_n\}$. Referring to our construction of λ , at least the $n - (2k-2) = n - 2k + 2 = \chi(\text{KG}(n, k))$ highest of these labels were assigned by Case II. In view of Tucker-Ky Fan's lemma, as $m = 2n-2k-9$, we can deduce that there exist at least $2k+8$ pairs of vertices of σ such that for any pair $\{v_i, v_j\}$ we have $|\lambda(v_i) + \lambda(v_j)| = 1$. Therefore, there exist two vertices $v_i, v_j \in V(\sigma)$ with $|\lambda(v_i)| \geq 2k-1$, $|\lambda(v_j)| \geq 2k-1$, and $|\lambda(v_i) + \lambda(v_j)| = 1$ such that $|P(v_i)| + |N(v_j)| \geq 2k+9$ or $|P(v_j)| + |N(v_i)| \geq 2k+9$. Without loss of generality, assume that $P(v_i) > N(v_i)$, and consequently, $|P(v_i)| + |N(v_j)| \geq 2k+9$. In view of definition of λ and since c is a star-free coloring and $|\lambda(v_i) + \lambda(v_j)| = 1$, one can conclude that all of k -subsets of $P(v_i)$ and $N(v_j)$ receive distinct colors. Hence, considering $n \leq 2k^3 - 2k^2 - 2k + 3$, we should have

$$\binom{|P(v_i)|}{k} + \binom{|N(v_j)|}{k} \leq 2\chi(\text{KG}(n, k)) - 11 \leq 4k^3 - 4k^2 - 8k - 1.$$

On the other hand, $|P(v_i)| + |N(v_j)| \geq 2k+9$ so

$$\binom{k+5}{k} + \binom{k+4}{k} \leq \binom{|P(v_i)|}{k} + \binom{|N(v_j)|}{k}.$$

But, one can check that for any $k \geq 2$, $\binom{k+5}{k} + \binom{k+4}{k} > 4k^3 - 4k^2 - 8k - 1$ which is a contradiction. ■

Remark 2. Similarly, one can show that if $n \geq 2k$ and $k \geq 81$, then $\chi_s(\text{KG}(n, k)) \geq \max\{2\chi(\text{KG}(n, k)) - 8, \chi(\text{KG}(n, k))\}$.

It seems that the star-free chromatic number of the Kneser graph $\text{KG}(n, k)$ is equal to $2\chi(\text{KG}(n, k)) - 2$ provided that $n \geq 2k \geq 4$.

Theorem 3. For any positive integers $n \geq 2k \geq 4$, if $n \leq \frac{8}{3}k$, then $\chi_s(\text{KG}(n, k)) = 2\chi(\text{KG}(n, k)) - 2 = 2n - 4k + 2$.

Proof. On the contrary, let $\chi_s(\text{KG}(n, k)) \leq 2\chi(\text{KG}(n, k)) - 3 = 2n - 4k + 1$ provided that $n \leq \frac{8}{3}k$. The proof is almost similar to that of Theorem 2. The labeling λ and $(n - 1)$ -simplex σ are defined similarly. As $m = 2n - 2k - 1$, we can deduce that there exist at least $2k$ pairs of vertices of σ such that for any pair $\{v_i, v_j\}$ we have $|\lambda(v_i) + \lambda(v_j)| = 1$. Therefore, there exist two vertices v_i and v_j with $|\lambda(v_i)| \geq 2k - 1$, $|\lambda(v_j)| \geq 2k - 1$, and $|\lambda(v_i) + \lambda(v_j)| = 1$ such that $|P(v_i)| + |N(v_j)| \geq 2k + 1$ or $|P(v_j)| + |N(v_i)| \geq 2k + 1$. Without loss of generality, assume that $P(v_i) > N(v_i)$, and consequently, $|P(v_i)| + |N(v_j)| \geq 2k + 1$. In view of definition of λ and since c is a star-free coloring, one can conclude that all of k -subsets of $P(v_i)$ and $N(v_j)$ receive distinct colors. Without loss of generality, suppose that $|P(v_i)| \geq k + 1$. For any vertex $v_j \in \sigma$ we have $P(v_i) \cap N(v_j) = \emptyset$. Consequently, in view of Tucker-Ky Fan's lemma, at least $\lfloor \frac{\chi(\text{KG}(n, k))}{2} \rfloor$ colors do not assign to the vertices of $\binom{P(v_i)}{k}$ which implies that

$$\binom{|P(v_i)|}{k} + \frac{\chi(\text{KG}(n, k)) - 1}{2} \leq 2\chi(\text{KG}(n, k)) - 3.$$

Therefore,

$$k + 1 \leq \frac{3}{2}\chi(\text{KG}(n, k)) - \frac{5}{2} = \frac{3}{2}(n - 2k + 2) - \frac{5}{2},$$

consequently, $n \geq \frac{8}{3}k + \frac{1}{3}$ which is a contradiction. ■

We know that $\chi_s(G) \leq \chi_l(G)$. Hence, we have the following corollary.

Corollary 2. For any positive integers $n \geq 2k \geq 4$, if $n \leq \frac{8}{3}k$, then $\chi_l(\text{KG}(n, k)) = 2\chi(\text{KG}(n, k)) - 2 = 2n - 4k + 2$.

Theorems 2 and 3 motivate us to propose the following conjecture which can be considered as a generalization of Kneser's conjecture.

Conjecture 1. For any positive integers $n \geq 2k \geq 4$, we have $\chi_s(\text{KG}(n, k)) = 2\chi(\text{KG}(n, k)) - 2 = 2n - 4k + 2$.

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